Recap

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- Given quiver $\vec{Q}=(I,H)$, $\alpha\in\mathbb{N}^I$ a dimension vector

$\rightsquigarrow E_\alpha/\mathbb{F}_q = \prod_{(i\to j)\in H}\mathrm{Hom}(\mathbb{A}^{\alpha_i},\mathbb{A}^{\alpha_j})$

$\rightsquigarrow$ have convolution functor

$$\star\; D^b_{c,G_\alpha}(E_\alpha)\times D^b_{c,G_\beta}(E_\beta)\longrightarrow D^b_{c,G_{\alpha+\beta}}(E_{\alpha+\beta})$$

$$(F_1^\bullet,F_2^\bullet)\longmapsto \lambda_!(r^*)^{-1}p^*(F_1^\bullet\boxtimes F_2^\bullet)[\dim E_{\alpha+\beta}]$$

- $e_i:=(0,\dots,1,\dots,0)\in\mathbb{N}^I$, $E_{e_i}=\{pt\}$, $\mathbb{1}_{e_i}=\underline{\mathbb{C}}_{E_{e_i}}$

$$\mathbb{L}=\langle \mathbb{1}_{\alpha_1}\star\cdots\star\mathbb{1}_{\alpha_n} \mid \alpha_i=e_{k_i}\text{ for some } k_i\in I\rangle$$

$$\underline{H}_\gamma=\langle \mathbb{1}_{\alpha_1}\star\cdots\star\mathbb{1}_{\alpha_n}\in\mathbb{L}\rangle_{\oplus,[1],\underline{\oplus}},\quad H_{\vec{Q}}=\coprod_{\gamma\in\mathbb{N}^I}\underline{H}_\gamma$$

- $L_{\alpha_1,\dots,\alpha_n}=\mathbb{1}_{\alpha_1}\star\cdots\star\mathbb{1}_{\alpha_n}$

$$=\lambda_!(\underline{\mathbb{C}}[\dim E_{\alpha_1,\dots,\alpha_n}])$$

- $L_{e_1}^{\star n}=[n]!\,\underline{\mathbb{C}}_{E_n}=[n]!\,L_{ne_1}$
- $L_{e_1}\star L_{e_2}=L_{e_2}\star L_{e_1}$
- $L_{e_1,e_1,e_2}\oplus L_{e_2,e_1,e_1}\cong[2]\,L_{e_1,e_2,e_1}$

<u>Prop</u>: Let $\vec{Q}= \overset{1}{\bullet}\rightrightarrows\overset{2}{\bullet}$ have r arrows in any direction. Then

$$\bigoplus_{\substack{i=0\\ i\text{ even}}}^{1+r}\begin{bmatrix}1+r\\ i\end{bmatrix}_q L_{e_1}^{\star i}\star L_{e_2}\star L_{e_1}^{\star(r+1-i)}\cong\bigoplus_{\substack{j=0\\ j\text{ odd}}}^{1+r}\begin{bmatrix}1+r\\ j\end{bmatrix}_q L_{e_1}^{\star j}\star L_{e_2}\star L_{e_1}^{\star(r+1-j)}$$

<u>Def</u> Let S_γ be the set of simple objects in $\underline{H}_\gamma$ and set $S_{\vec{Q}}=\coprod_{\gamma\in\mathbb{N}^I}S_\gamma$

Let $K_{\vec{Q}}=\bigoplus_{\gamma\in\mathbb{N}^I}K_\gamma=\bigoplus_{\gamma\in\mathbb{N}^I}K_0(\underline{H}_\gamma)$. Note

$$K_\gamma=\bigoplus_{\substack{F^\bullet\in S_\gamma\\ n\in\mathbb{Z}}}\mathbb{Z}[F^\bullet[n]]=\bigoplus_{F^\bullet\in S_\gamma}\mathbb{Z}[q,q^{-1}][F^\bullet]$$

Main Theorem of Lusztig

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- $\star$ preserves $H_{\vec{Q}} \Rightarrow K_{\vec{Q}}$ has mult

<u>Def</u> Given a lie algebra $\mathfrak{g}$, the integral form of $U_q(\mathfrak{g})$ is the subalgebra $U_q^{\mathbb{Z}}(\mathfrak{g})$ gen by

$$E_i^{(n)} = \frac{E_i}{[n]!}, \quad F_i^{(n)} = \frac{F_i}{[n]!}, \quad K_i^{\pm 1}, \; i \in \Delta^+$$

<u>Main thrm:</u> Let $\vec{Q}$ be a quiver w/o loops. Let $\mathfrak{g}_{\vec{Q}}$ be Kac-Moody lie alg associated to $\vec{Q}$, and $n_{\vec{Q}}^+$ = positive part of $\mathfrak{g}_{\vec{Q}}$. Then the map

$$\Phi : U_q^{\mathbb{Z}}(n_{\vec{Q}}^+) \longrightarrow K_{\vec{Q}}$$
$$E_i^{(n)} \longmapsto [\mathbb{1}_{ne_i}]$$
$$q \longmapsto q^{-1}$$

is an isomorphism of algs

<u>Pf:</u> (When $\vec{Q}$ is finite type = ADE)

Fact: For $\vec{Q}$ finite type, $S_\gamma = \{ IC(\mathcal{O}) \mid \mathcal{O} \in E_\gamma / G_\gamma \}$

Know Φ is a homomorphism from prev calc

<u>Step 1:</u> Both $U_q^{\mathbb{Z}}(n_{\vec{Q}}^+)$, $K_{\vec{Q}}$ free / $\mathbb{Z}[q^{\pm 1}]$

$\Rightarrow$ Show graded ranks agree.

Let Φ^+ = positive roots of $\mathfrak{g}$.

$\cup$

Δ^+ = simple roots of Φ^+ $(r = |\Delta^+|)$

Given $\gamma \in \mathbb{Z}\Delta^+$,

$$U_q^{\mathbb{Z}}(n_{\vec{Q}}^+)_\gamma \overset{PBW}{\cong} \operatorname{span}\left\{ E_{\alpha_1}^{(n_1)} \cdots E_{\alpha_m}^{(n_m)} \;\middle|\; \begin{array}{c} \sum n_j \alpha_j = \gamma \\ \alpha_j \in \Phi^+ \end{array} \right\}$$

Write $\gamma = \sum_{i=1}^r c_i \beta_i$, $\beta_i \in \Delta^+$

Let $\gamma' = \sum_{i=1}^r c_i e_i \in \mathbb{N}I$

$$\Phi : U_q^{\mathbb{Z}}(n_{\vec{Q}}^+)_\gamma \to K_{\gamma'}$$

By construction $K_{\gamma'}$ has rank

$$\operatorname{rk} K_{\gamma'} = \# |S_{\gamma'}| = \# | E_{\gamma'} / G_{\gamma'} |$$

Main Theorem Proof

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Obs

$$\left\{ G_{\gamma'} \text{ orbits in } E_{\gamma'} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{isomorphism classes of} \\ \mathrm{Rep}_{\mathbb{F}_q}(\vec{Q}) \text{ of dim } \gamma' \end{array} \right\}$$

- B/c $\mathrm{Rep}_{\overline{\mathbb{F}_q}}(\vec{Q})$ is Krull-Schmit

$$\left\{ \begin{array}{l} \text{isomorphism classes} \\ \text{of } \mathrm{Rep}_{\overline{\mathbb{F}_q}}(\vec{Q}) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \oplus \text{ of indecomp} \\ \text{w/ multiplicites} \end{array} \right\}$$

indecomp

$$\Rightarrow \mathrm{rk}\, K_{\gamma'} = \#\left\{ \oplus \underline{I_k}^{m_k} \,\middle|\, \sum m_k \dim \underline{I_k} = \gamma' \right\}$$

Thrm (Gabriel) Let $\vec{Q}$ be finite type. Then the map $\mathrm{Ind}\,\vec{Q} \longrightarrow \Phi^+$

$$\underline{I_k} \to \underline{\dim I_k} \xrightarrow{e_i \mapsto \alpha_i} \underline{\dim' I_k}$$

is a bijection.

Cor: $\mathrm{rk}\, U_q^{\mathbb{Z}}(n^+)_\gamma = \mathrm{rk}\, K_{\gamma'}$

- $\mathrm{rk}\, K_{\gamma'}$ is finite, thus suffices to show Φ is surjective. Roughly we use

(1) **Thrm:** Let $\vec{Q}$ be finite type quiver, γ dim vector. For any orbit $\mathcal{O} \subset E_\gamma$ $\exists$ seq $(c_1 e_{i_1}, \dots, c_n e_{i_n})$, $\sum c_k e_{i_k} = \gamma$ s.t.

$$E_{c_1 e_{i_1}, \dots, c_n e_{i_n}} \longrightarrow E_\gamma$$

is a resol of singularties over $\overline{\mathcal{O}}$

(2) Spam Decomposition Thrm ∎

Rem: Can extend iso $\Phi: U_q^{\mathbb{Z}}(n^+_{\vec{Q}}) \simeq K_{\vec{Q}}$ to isometry of Hopf algs

Def Let $F^\bullet, G^\bullet \in \mathcal{H}_\gamma$. Then let

$$\{F^\bullet, G^\bullet\} = \sum_j \left(\dim H^j_{G_\gamma}(E_\gamma, F^\bullet \overset{L}{\otimes} G^\bullet)\right) q^j$$

- $\{,\}$ descends to K_γ due to LES in cohomology

Canonical Basis

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Prop 1. $\{,\}$ is symm, $\mathbb{Z}[q,q^{-1}]$ bilinear

2. $\{F^\bullet \star G^\bullet, H^\bullet\} = \{F^\bullet \boxtimes G^\bullet, \Delta(H^\bullet)\}$

3. If $S_1^\bullet, S_2^\bullet$ are simple perverse sheaves

$$\{S_1^\bullet, S_2^\bullet\} \in \begin{cases} 1 + q\mathbb{N}[q] & \text{if } S_2 \simeq \mathbb{D}(S_1^\bullet) \\ q\mathbb{N}[q] & \text{otherwise} \end{cases}$$

Exer: Prove 3 $\Rightarrow$ $\{,\}$ is non-deg on K_γ

Def The canonical basis of $U_q^\mathbb{Z}(n^+_{\vec{Q}})$ is

$$B_{\vec{Q}} = \{\Phi^{-1}(S^\bullet) \mid S^\bullet \in S_{\vec{Q}}\}$$

under iso $\Phi : U_q^\mathbb{Z}(n^+_{\vec{Q}}) \simeq K_{\vec{Q}}$

What about $B_{\vec{Q}}$ is canonical?

Prop: $B_{\vec{Q}}$ is independent of the choice of orientation for $\vec{Q}$

Sketch of Pf: Let $\vec{Q}'$ have different orientation. Then $\exists$ equivalence

$$\Theta : D^b(E_\gamma) \xrightarrow{\sim} D^b(E'_\gamma)$$

that restricts to $\underline{H}_\gamma \simeq \underline{H}'_\gamma$ and gives bijection $S_\gamma \longleftrightarrow S'_\gamma$, and $\theta(\mathbb{1}_{ne_i}) = \mathbb{1}_{ne_i}$ ✓

$$\begin{tikzcd} U_q^\mathbb{Z}(n^+) \arrow[r, "\Phi"] \arrow[d, "id"'] & K_\gamma \arrow[d, "{[\theta]}"] \\ U_q^\mathbb{Z}(n^+) \arrow[r, "\Phi'"] & K'_\gamma \end{tikzcd}$$

$$B'_{\vec{Q}} = \Phi'^{-1}(S'_\gamma) = \Phi'^{-1}(\theta(S_\gamma)) = \Phi^{-1}(S_\gamma) = B_{\vec{Q}}$$

Properties of the Canonical Basis

Prop Let λ be an integral dominant weight of $\mathfrak{g}_{\vec{Q}}$. Let $V(\lambda)$ be corresponding integrable highest weight rep of $U_q^{\mathbb{Z}}(\mathfrak{g}_{\vec{Q}})$ w/ highest weight vector v_λ. Then

$$B_\lambda := \{ b \cdot v_\lambda \mid b \in B_{\vec{Q}},\ b \cdot v_\lambda \neq 0 \}$$

forms a weight basis of $V(\lambda)$

Sketch Pf: h.w $\Rightarrow V(\lambda) = U_q^{\mathbb{Z}}(\mathfrak{b}^-_{\vec{Q}}) / I_\lambda$

and have basis in $U_q^{\mathbb{Z}}(\mathfrak{b}^-_{\vec{Q}})$

- Content of prop is that projection of basis is L.I. $\boxtimes$

Rmrk: Can specialize $U_q^{\mathbb{Z}}(\mathfrak{g}_{\vec{Q}})$ at $q=1$ to obtain canonical basis for integrable $\mathfrak{g}$-mod

Rmrk: Can construct can basis for $\otimes$ of int modules using algor very similar to Id basis

More Properties of $B_{\vec{Q}}$

(1) $b \cdot b' \in \bigoplus \mathbb{N}[q, q^{-1}]\, b''$

$b'' \in B_{\vec{Q}}$

(2) same w/ coproduct

(3) $\langle\ ,\ \rangle := \{ \mathcal{I}-, \mathcal{I}- \}$,

$\langle b, b \rangle \in 1 + q^{-1}\mathbb{N}[q^{-1}]$

$\langle b, b' \rangle \in q^{-1}\mathbb{N}[q^{-1}]$

(4) Let $\bar{\ } : U_q^{\mathbb{Z}}(\mathfrak{b}^+) \to U_q^{\mathbb{Z}}(\mathfrak{b}^+)$

$E_i^{(n)} \longmapsto E_i^{(n)}$

$K_i \longmapsto K_i^{-1}$

$q \longmapsto q^{-1}$

Then $\bar{b} = b$ $\quad (\bar{\ } \longleftrightarrow \mathbb{D})$

Thrm: Let $\mathcal{B}$ = set of all elements $b \in U_q^{\mathbb{Z}}(\mathfrak{b}^+)$ s.t. $\bar{b} = b$, $\langle b, b \rangle \in 1 + q^{-1}\mathbb{N}[q^{-1}]$

Then $\mathcal{B} = B_{\vec{Q}} \cup -B_{\vec{Q}}$

Connection with KL

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Recall quantum Schur Weyl duality

Let $V = \mathbb{C}(q)^{\oplus k}$

$$U_q(\mathfrak{sl}_k) \curvearrowright V^{\otimes n} \curvearrowleft H(S_n) = H$$

$$CB = pKL$$

Rmrk: wt spaces of $V^{\otimes n}$ isomorphic to

$$V^{\otimes n}[\underline{d}] \cong H \otimes_{H_{\underline{d}}} \mathbb{C} \quad (q \to -1)$$

as H-mod. RHS has pKL basis constructed from H-action.

Rmrk: We saw this isom in the form

$$V^{\otimes n}[\underline{d}] \cong \bigoplus_{\substack{\underline{d} \in (\mathbb{Z}_{\geq 0})^k \\ \sum d_i = n}} \left[\mathcal{O}(\mathfrak{gl}_n)_{\underline{d}}\right]$$

sing block

where $k = 2$ last semester

There, the possible parabolic subgroups were $S_i \times S_{n-i}$

Ex: $\vec{Q} = \bullet$, $B_{\vec{Q}} = \{E_1^{(n)} \mid n \in \mathbb{N}\}$

Ex: For $\vec{Q} = \underset{1}{\bullet}\!-\!\!-\!\underset{2}{\bullet}$, $B_{\vec{Q}}$ is

$$\left\{ E_1^{(a)} E_2^{(b)} E_1^{(c)},\ E_2^{(c)} E_1^{(b)} E_2^{(a)} \;\middle|\; \begin{array}{l} b \geq a+c, \\ a,b,c \in \mathbb{N} \end{array} \right.$$

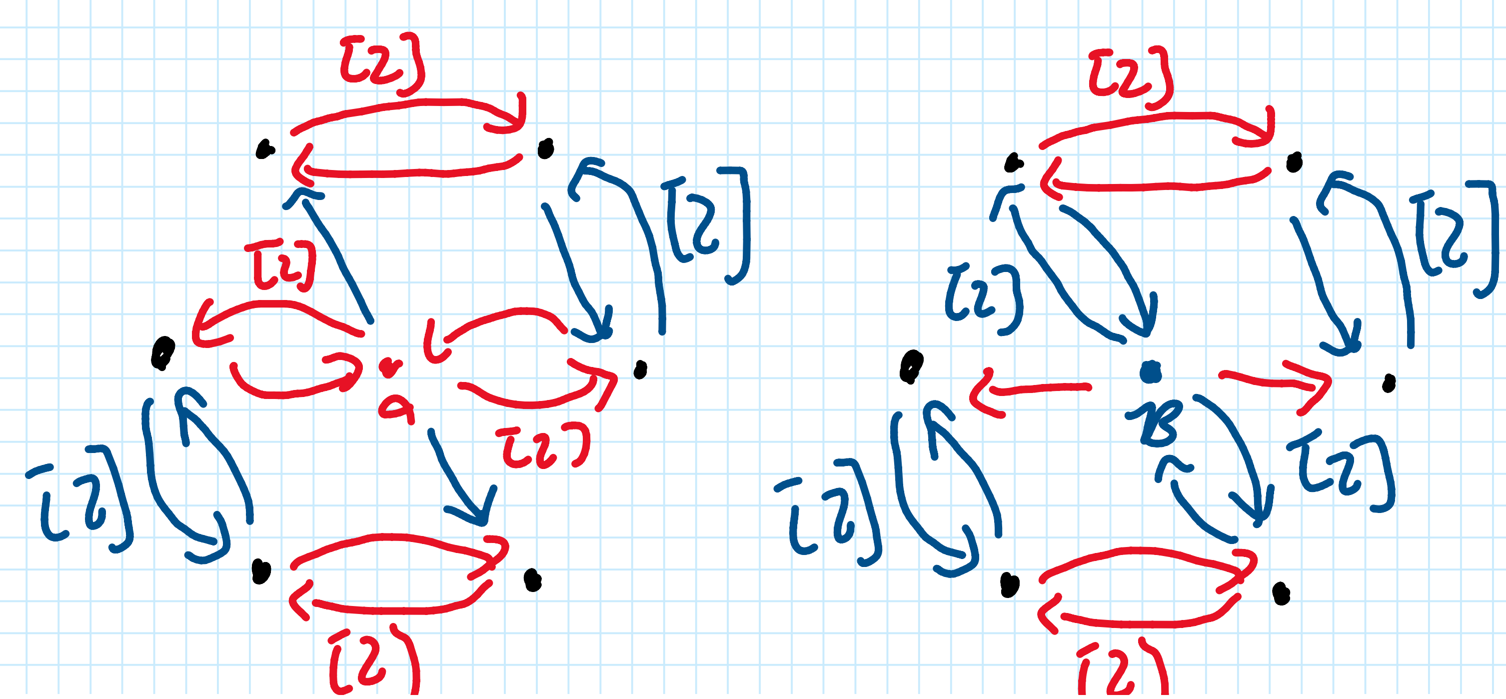
w/ identification

$$\left. E_1^{(a)} E_2^{(b)} E_1^{(b-a)} = E_2^{(b-a)} E_1^{(b)} E_2^{(a)} \right\}$$

Ex: quantum adjoint rep of $U_q(\mathfrak{sl}_3)$

$\to := \xrightarrow{[1]}$, $\beta \nwarrow \ \to \alpha$, $\to = E_\alpha$, $\to = E_\beta$

$\leftarrow = F_\alpha$, $\leftarrow = F_\beta$



were six Sn-i

(2) (2)

Crystal Basis

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Informally a crystal basis for $U_q(n^+_{\vec{Q}})$ is a basis at "$q=0$" s.t. action of $E_i^{(k)}, F_i^{(k)}$ is nice

- It turns out if you have a crystal basis at "$q=0$" and "$q=\infty$" you can lift in a unique way to get an actual basis $G_{\vec{Q}}(\infty)$ for $U_q(n^+_{\vec{Q}})$ called the global basis.

Thrm $B_{\vec{Q}} = G_{\vec{Q}}(\infty)$

SLOGAN: Character multiplicties come from canonical bases

Ex1 ($osp(2m+1|2n)$)

Q: qSchur-Weyl Duality for type B

$??? \circlearrowright V^{\otimes n} \circlearrowleft H(B_n)$

A: ??? = quantum symmetric pair $= (U, U^i)$

Thrm (Bao, Wang-16) (U, U^i) has an i-canonical basis

Thrm (Bao, Wang-18): Roughly states

$$L(\lambda) = \sum Q_{\lambda\mu}(1)\, M(\mu)$$

- $L(\lambda)$ = simple in $osp(2m+1|2n)$
- $M(\lambda)$ = Verma in $osp(2m+1|2n)$
- $Q_{\lambda,\mu}(q)$ = i-KL poly in $V^{\otimes m} \otimes V^{*\otimes n}$ = transition matrix between i-canonical basis and standard monomial basis in $V^{\otimes m} \otimes V^{*\otimes n}$

LLT conjecture

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Ex 2 (LLT) - $H(S_n)/\mathbb{C}(q)$ is s.s. w/ irreducibles given by Specht modules $S(\lambda)$, $\lambda \vdash n$

- Let $H_d(S_n)/\mathbb{C} = H(S_n)|_{q=\xi^{2d}}$, ξ primitive 2d root of unity
- $S(\lambda)$ no longer irr,
  $L(\lambda)$ = irr $H_d(S_n)$-mod corr to λ

LLT conjecture;

$$[S(\lambda):L(\mu)] = d_{\lambda,\mu} \text{ where}$$

- $P_\lambda = \sum_{\mu \vdash n} d_{\lambda\mu} [\lambda]$
- B_λ = canonical basis in $V(\Lambda_0)$
- $V(\Lambda_0)$ = irr h.w $U(\widehat{sl_d})$ mod in Fock space F gen by $[\phi]/1$
- $F = \bigoplus_{\lambda \text{ a partition}} \mathbb{C}[\lambda] = \mathbb{C}[b_{-1}, b_{-2}, \ldots]$

Rmrk: LLT conjecture proved in 96 by Ariki. $\exists$ q-analog proved in 09 by [BK] using KLR algebras

Let $\mathbb{L}_{\vec{Q}} = \bigoplus_{S^\cdot \in S_{\vec{Q}}} S^\cdot \in \underline{H}_{\vec{Q}}$. By def $\langle \mathbb{L}_{\vec{Q}} \rangle_\Delta = \underline{H}_{\vec{Q}}$. By "dg-Morita theory"

$$\mathcal{H}om(\mathbb{L}_{\vec{Q}}, -) : \underline{H}_{\vec{Q}} \xrightarrow{\sim} D(dg\text{-}End(\mathbb{L}))$$

$$U_q\mathfrak{Z}(n^+_{\vec{Q}}) = K_0(\underline{H}_{\vec{Q}}) = K_0(\mathbb{L})$$
$$= K_0(dg\text{-}H^\cdot(End(\mathbb{L}_{\vec{Q}})))$$
$$= K_0(Ext^\cdot(\mathbb{L}_{\vec{Q}})\text{-}gmod)$$

Def $Ext^\cdot(\mathbb{L}_{\vec{Q}})$ is KLR alg associated to $\vec{Q}$

Historically, this isn't the right def

KLR Algebras

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Thrm (Varagnolo, Vasserot): Let $\vec{Q}$ be simply laced. Let $R_{\vec{Q}}$ be KLR alg in sense of [KL], [R]. Then

$$R_{\vec{Q}} \cong \operatorname{Ext}^{\bullet}(\mathbb{1}_{\vec{Q}})$$

$$K_{\oplus}(R_{\vec{Q}}\text{-gmod}) \cong U_q^{\mathbb{Z}}(n^+_{\vec{Q}})$$

$$[\text{indecomp proj}] \longleftrightarrow \text{canonical basis}$$

$$[\text{simples}] \longleftrightarrow \text{dual canonical basis}$$

Warning: For $\vec{Q}$ not simply laced,

$$[\text{indecomp proj}] \neq \text{canonical basis}$$